Linear systems – Final exam – Version B

Final exam 2019–2020, Tuesday 16 June 2020, 8:30 – 12:00

Instructions

1. The exam is **open book**, meaning that the use of the lecture notes and other course material (everything you can find on the Nestor page of the course) is allowed. You are also allowed to use your own handwritten notes.
2. The use of sources other than described above, such as books or web pages, is **not** allowed.
3. All answers need to be accompanied with an explanation or calculation.
4. Please clearly indicate the exam version on your exam paper.

Problem 1 (5 + 5 + 8 + 4 + 6 = 28 points)

A model for the spread of the Corona virus in a population is given as

$$\begin{aligned} \dot{s}(t) &= -\beta s(t)q(t), \\ \dot{q}(t) &= \beta s(t)q(t) - \gamma q(t), \\ \dot{r}(t) &= \gamma q(t), \end{aligned} \tag{1}$$

where $s(t) \in \mathbb{R}$, $q(t) \in \mathbb{R}$, and $r(t) \in \mathbb{R}$ are respectively the fraction of susceptible, infected, and recovered individuals. The parameter $\beta > 0$ represents the rate of infection, whereas $\gamma > 0$ is the rate of recovery.

Consider the initial value problem with dynamics (1) and initial condition $(s(0), q(0), r(0)) = (s_0, q_0, r_0)$ satisfying

$$s_0 + q_0 + r_0 = 1. \tag{2}$$

(a) Show that the corresponding solution satisfies $s(t) + q(t) + r(t) = 1$ for all $t \geq 0$.

In the remainder of this problem, assume in addition to (2) that $s_0 > 0$, $q_0 > 0$, $r_0 \geq 0$, which can be shown to imply $s(t) > 0$, $q(t) > 0$, $r(t) \geq 0$ for all $t \geq 0$.

(b) We will relate $s(\cdot)$ and $r(\cdot)$ by looking for a function $S : \mathbb{R} \to \mathbb{R}$ such that

$$s(t) = S(r(t))$$

for all $t \geq 0$. Show that S satisfies

$$\frac{\mathrm{d}S}{\mathrm{d}r} = -\frac{\beta}{\gamma} S. \tag{3}$$

(c) Use (3) to show that the fraction of susceptible and recovered individuals satisfies

$$s(t) = s_0 e^{-\frac{\beta}{\gamma}(r(t) - r_0)}.$$

for all $t \geq 0$.

(d) It can be shown that

$$\lim_{t \to \infty} \big(s(t), q(t), r(t) \big) = (s_\infty, 0, r_\infty),$$

for some s_∞, r_∞. Show that $(s_\infty, 0, r_\infty)$ is an equilibrium of the system (1).

(e) Linearize the system (1) around the equilibrium $(s_\infty, 0, r_\infty)$.

Problem 2 (14 points)

Consider the linear system $\dot{x}(t) = Ax(t)$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b & -3b & -a & -2a \end{bmatrix}$$

where $a, b \in \mathbb{R}$. Determine all values of a and b for which the system is asymptotically stable.

Problem 3 ($4 + 12 + 8 = 24$ points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t),$$

with $x(t) \in \mathbb{R}^2$, $y(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 3 \end{bmatrix}.$$

(a) Is the system observable?

(b) Find a nonsingular matrix T and real scalars α_1, α_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & \alpha_1 \\ 1 & \alpha_2 \end{bmatrix}, \qquad CT^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

(c) Use the matrix T from problem (b) to obtain a stable state observer of the form

$$\dot{\xi}(t) = A\xi(t) + Bu(t) + G(y(t) - C\xi(t)),$$

where G is chosen such that $A - GC$ has eigenvalues -2 and -3.

Problem 4 ($5 + 5 = 10$ points)

Consider the linear system

$$\dot{x}(t) = \begin{bmatrix} -1 & -4 & 0 \\ 0 & 3 & 0 \\ 6 & 6 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} u(t).$$

(a) Is the system controllable? If not, give a basis for the reachable subspace.

(b) Is the system stabilizable?

Problem 5 (14 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \tag{4}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$. Denote its reachable subspace by $\mathcal{W}$ and its unobservable subspace by $\mathcal{N}$.

Assume that the impulse response matrix satisfies $H(t) = 0$ for all $t \in \mathbb{R}$. Show that $\mathcal{W} \subset \mathcal{N}$.

(10 points free)